Beyond A + B = C
Building Algebraic Thinking in the Early Grades

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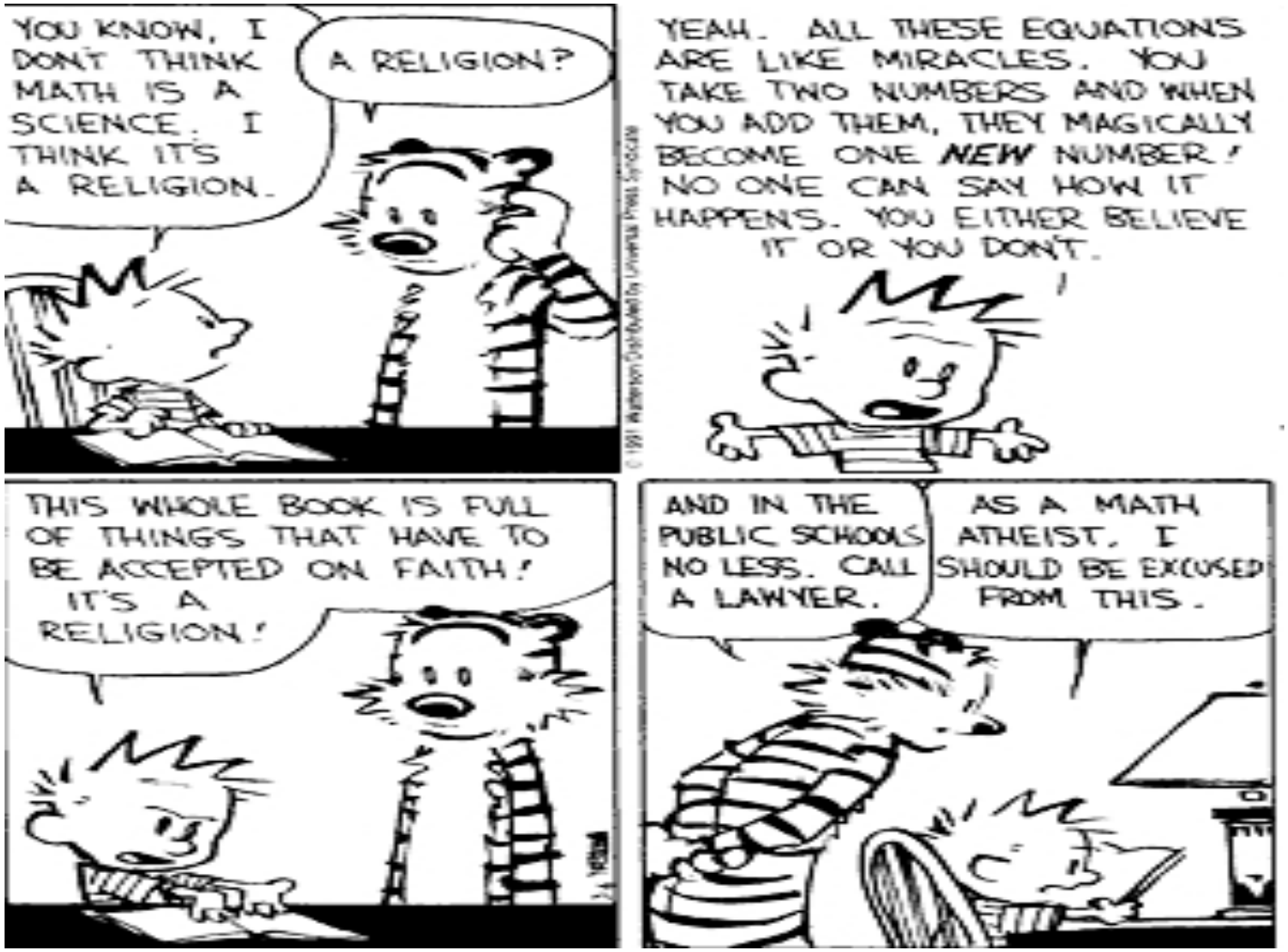
The Big Ideas

- Flexible Thinking Across Contexts
  - Relational Thinking
  - Thinking Deeply about Arithmetic
    - The Arithmetic and Algebra Connection
  - Multiple Problem Contexts

Making Math Less Magical



What does it mean to be a flexible thinker?

- Thinking flexibly about number
- Interpreting equations in varied formats
- Understanding of equality
- Ability to interpret a variety of problem types
- Ability to solve problems in a variety of contexts

Why Relational Thinking?

- Relational thinking is understanding the relationships in equations
- The key to Algebra is relational thinking
- Relational thinking helps students deepen their understanding of and flexibility with arithmetic
- Relational thinking is a building block of middle and high school algebra courses

Relational Thinking Deepens Procedural Understanding

- Relational thinking helps students think more flexibly about number
- Aids in the mastery of addition, subtraction, multiplication and division facts
- Eases the transition to multi-digit operations

Why vary problem types?

- Explore the various applications of operations
- Increase ability to interpret problem contexts

How would your students respond to the following question?

What number would you put in the box to make this a true number sentence?

$$8 + 4 = \blacksquare + 5$$

Video – Students Struggling with Equality

- In the following video, we are going to see a group of students coming to an understanding about the equal sign.
- As you watch, think about what surprises you about the video and what you see that you expected to see.

Video – Students Struggling with Equality

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School. Portsmouth, NH: Heinemann.*

Kids Are Confused About The Equal Sign

- The equal sign is misunderstood to be an operator
- Children misunderstand equality in various forms (e.g. 6 = 6, 5 = 2 + 3)
- Children often struggle to solve equations like

$$8 + 4 = \blacksquare + 5$$

What can we do to help students develop this understanding?

- Represent number sentences in different forms

  $a + b = c$

  $c = c$

  $c = a + b$

  $a + d = c + b$

- Use the equal sign correctly at all times

- Use terminology like "is the same as" instead of "is equal to" when reading "="

How do children come to understand the concept of Equality and the meaning of the equal sign?

- Carpenter (2003) believes that children come to an understanding by moving through four benchmark stages of development.
- This understanding develops differently in different children
- It is not a linear process and children often skip steps

Carpenter's Benchmarks for Understanding the Equal Sign

- Children must be specific about what they think the equal sign means
- Children must accept as true some number sentence that is not of the form a + b = c
- Children must demonstrate understanding that the equal sign represents a relationship between two equal numbers
- Children are able to compare the mathematical expressions on either side of the equal sign without carrying out the operations

How can you push your students forward in their understanding?

- First, you need to establish where they are in their understanding.
- Once you establish where they are, you can help them to develop their understanding further.
- Where do you think your students are in this understanding?

Children must be specific about what they think the equal sign means

I asked a group of 8th graders to "define" the equal sign. Here are some of their responses . . . .

"The equal sign is a sign to find the answer"

"The equal sign means that a problem is simplified as much as possible"

"It shows that one thing is the same amount as another thing"

"What the answer is, the sum or product of a problem

"In my mind, the equal sign means the sum of two digits"

"The sum, product, quotient, difference of any number. It's the answer to a math problem"

"The equal sign represents the balance between two or more numbers that have the same equivalency"

"The equal sign means that there are two values that are the same"

"Means that one side of the equation is neither greater than or less than. They are the same, although one side may look very different than the other."

Children must accept as true some number sentence that is not of the form a + b = c

- Which number sentences are true and which sentences are false?
- Be prepared to give a reason for your answer

$7 + 6 = 13$

$13 = 13$

$13 = 7 + 6$

$7 + 6 = 13 + 0$

$13 + 0 = 13$

$7 + 6 = 6 + 7$

Children must demonstrate understanding that the equal sign represents a relationship between two equal numbers.

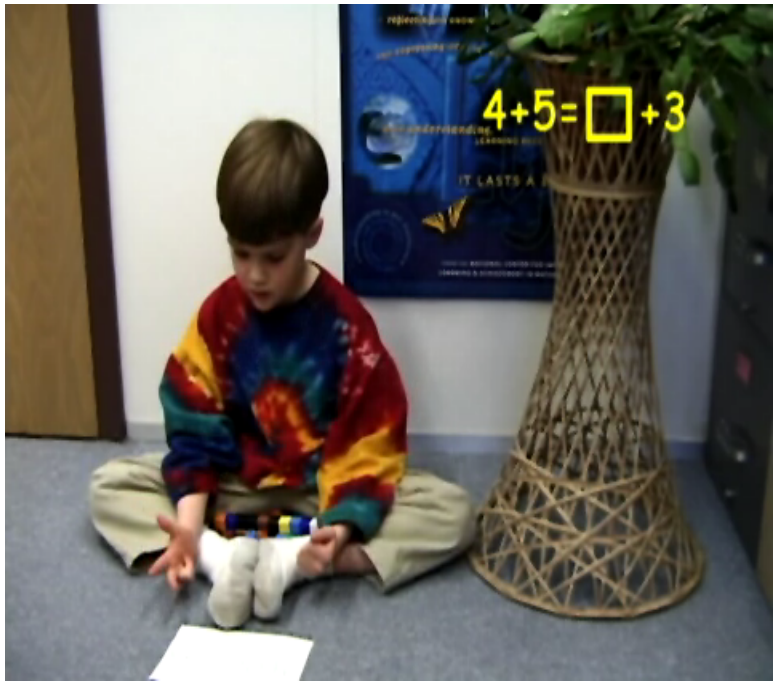
They come to understand that a sentence like

8 + 3 = 7 + 4

is true because

8 + 3 = 11 and 7 + 4 = 11

and 11 = 11



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Children are able to compare the mathematical expressions on either side of the equal sign without carrying out the operations

3 + 5 = ■ + 6

Without using computation to solve, articulate your strategy for solving this problem to a neighbor

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Test Your Relational Thinking

$$35 + 47 = 37 + \blacksquare$$

$$16 + \blacksquare = 20 + 15$$

$$157 + 321 = \blacksquare + 331$$

$$\blacksquare + 765 = 766 + 261$$

$$817 + \blacksquare = 989 + 807$$

Using Relational Thinking to Master the Basic Facts

- Relational thinking can help students to master their basic facts
  - Adding 9's
  - Doubling + 1
  - Doubling – 1
  - Building 10's
  - Building 5's

Problem Types

- Students who are able to execute relational thinking are better prepared to handle a variety of problem types
- Addition and subtraction can be presented in a variety of contexts and problem situations, each of which could be represented with a different equation – where the unknown is placed in various places in the equation.
- Multiplication and division can also be presented in a variety of contexts, represented with a variety of equations, with unknowns on either side of the equal sign.

Write an equation to represent each problem situation below

Two birds sat on a fence. Three more birds flew over to join them. How many birds are on the fence now?

2 + 3 = ?

Result unknown

Two birds were sitting on the fence. Some more birds flew over. Then there were five birds. How many birds flew over?

2 + ? = 5

Change unknown

Some birds were sitting on the fence. Three more birds flew over. Then there were five birds on the fence. How many birds were on the fence to start?

? + 3 = 5

Start unknown

Write an equation to represent each problem situation below

Five birds sat on a fence. Two birds flew away. How many birds are on the fence now?

5 - 2 = ?

Result unknown

Five birds were sitting on the fence. Some birds flew away. Then there were 3 birds sitting on the fence. How many birds flew away?

5 - ? = 3

Change unknown

Some birds were sitting on the fence. Two birds flew away. Then there were three birds on the fence. How many birds were on the fence to start?

? - 2 = 3

Start unknown

Write an equation to represent each problem situation below

Three birds and two squirrels are on the fence. How many animals are on the fence?

$2 + 3 = ?$

Total unknown

Five animals are sitting on the fence. 2 are birds and some are squirrels. How many squirrels are sitting on the fence?

$2 + ? = 5$
$5 - 2 = ?$

Addend unknown

There are five animals sitting on the fence. Some are birds and some are squirrels. How many birds and squirrels are there?

$5 = 0 + 5,\ 5 = 5 + 0,$
$5 = 4 + 1,\ 5 = 1 + 4,$
$5 = 2 + 3,\ 5 = 3 + 2$

Both addends unknown

Write an equation to represent each problem situation below

Lucy has 2 apples. Julie has 5 apples. How many more does Julie have than Lucy?

OR

How many fewer does Lucy have than Julie?

2 + ? = 5
5 - ? = 2

Difference unknown

Julie has 3 more apples than Lucy. Lucy has 2 apples. How many apples does Julie have?

OR

Julie has 3 less apples than Lucy. Julie has 2 apples. How many apples does Lucy have?

2 + 3 = ?
3 + 2 = ?

Bigger unknown

Julie has 3 more apples than Lucy. Julie has 5 apples. How many apples does Lucy have?

OR

Lucy has 3 fewer apples than Julie. Julie has 5 apples. How many apples does Lucy have?

5 – 3 = ?
? + 3 = 5

Smaller unknown

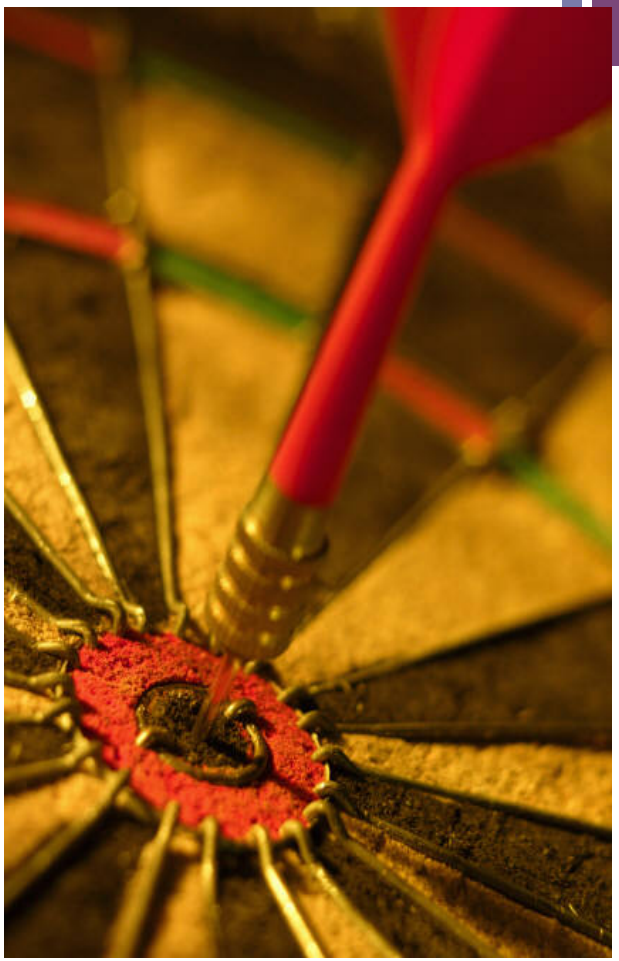
	Unknown Product $3 \times 6 = ?$	Group Size Unknown ("How many in each group?" Division) $3 \times ? = 18$ and $18 \div 3 = ?$	Number of Groups Unknown ("How many groups?" Division) $? \times 6 = 18$ and $18 \div 6 = ?$
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example.* You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example.* You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example.* You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays,[131] Area[132]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example.* What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example.* A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example.* A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example.* A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs $18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example.* A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat? *Measurement example.* A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	$a \times b = ?$	$a \times ? = p$ and $p \div a = ?$	$? \times b = p$ and $p \div b = ?$

Summing Up

- Relational thinking is a key to algebra AND can provide a means for children to truly understand arithmetic.
- The equal sign is the most misunderstood symbol in mathematics
- Students move through a series of benchmarks to a relational understanding of equality

Target Number

- First, we choose a target number
- Then, we choose 5 numbers from the digit deck
- Now, using all operations, make as many true sentences that are equal to the target number.
- The more digits you use, the more points you earn.



Target Number: Trial Run

- The target number is 10
- The digits selected are 3, 7, 4, 2, 6
- Try to make equations where both sides are equivalent to 10 using as many operations as possible.
- $3 + 7 = 10$ (earns you 2 points)
- $2 \times 7 - 4 = 10$ (earns you 3 points)
- $10 = 2 \times 3 + 4$ (earns you 3 points)
- $4 + 6 = 7 + 3$ (earns you 4 points)

Target Number
You play

- Let's select a target number
- Let's select 5 digit card
- Now, you play. The person who earns the most points wins!