*__INDIRECT PROOF__*:

Some students get the hang of indirect proof easily, but for others it seems a foreign and unnatural way of thinking. I think there are a number of problems that one can use to introduce proof by contradiction as something students are actually and easily doing.

<u>1) ***Use Magic Squares***</u>:

Consider the following problem: In a magic square the sum of the elements in the rows, columns, and diagonals are equal. That sum, called the magic sum equals one-third the sum of all the elements. Also, the number in the middle of a 3 by 3 magic square equals one-third the magic sum. If we try to arrange the numbers from 1 to 9 to form a magic square, the magic sum is 15 so we place a 5 in the middle square. Now the question is, where does the 9 go? Suppose we place 9 in a corner. Then 1 must be placed opposite 9 to obtain 15. This leaves 2 and 4 as the only numbers that can go in the same row as 9. But if we place them as shown in the 4th diagram from the left, we have to put a 12 between the 1 and the 2, but we can't use 12. If we place the 2 and 4 as shown in the rightmost diagram, then 6 must go in the lower left hand corner to make a sum of 15, but then there is no number that can go in the middle of the left hand column. So 9 can't go in a corner because that leads to impossible situations.

| | | |
|---|---|---|
| | | |
| | 5 | |
| | | |

| | | |
|---|---|---|
| 9 | | |
| | 5 | |
| | | |

| | | |
|---|---|---|
| 9 | | |
| | 5 | |
| | | 1 |

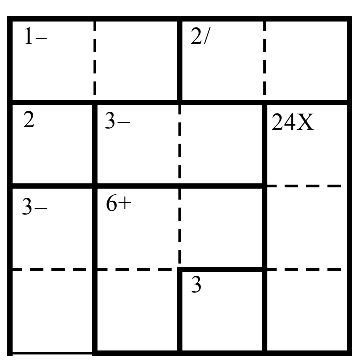
| | | |
|---|---|---|
| 9 | 4 | 2 |
| | 5 | |
| | | 1 |

| | | |
|---|---|---|
| 9 | 2 | 4 |
| | 5 | |
| | | 1 |

As the above example demonstrates, in indirect proof, one starts with an assumption and shows that the assumption leads to results that can't be.

<u>2) ***Use Kenken or Sudoko***</u>

Sudoko and Kenken work exactly this way. One tries putting various numbers in squares, discovers contradictions, realizes that those choices fail, and tries something else until the results have no contradictions. Here's a Kenken problem. The rules are: each row and column must contain the numbers from 1 to 4 without repeating. The numbers in the outlined boxes must combine using the given operation (in any order) to produce the target numbers in the top-left corners. Fill in single-box cages with the number in the top-left corner.



Consider the rectangle in the upper left. Two numbers with a difference of 1 go there. Let's try 2 and 1. If I put the 2 in the uppermost left, there is a conflict with the boxes right below which requires a 2. So I'll put a 1 in the upper most left and the 2 in the box to the right. The next rectangle over requires two numbers whose quotient is 2. The only possibilities are 1 and 2 or 2 and 4, but neither work because I've already used the 2. So I can't use 1 and 2 in the upper left rectangle. This sort of reasoning continues until we have the completed box below:

| | | | |
|---|---|---|---|
| 1− **3** | **4** | 2/ **2** | **1** |
| 2 **2** | 3− **1** | **4** | 24X **3** |
| 3− **4** | 6+ **3** | **1** | **2** |
| **1** | **2** | 3 **3** | **4** |

3) ***Use Riddles***

Riddles provide another source of reasoning indirectly that seems natural. Try to solve it before looking at the solution.

If exactly one of the following three people is telling the truth and exactly one person did it, who, in fact, did the dastardly deed.

Al: I did it.
Bob: I didn't do it.
Carl: Al didn't do it.

Solution: Start by assuming that each person in turn was the one telling the truth and see if a contradiction emerges.

If Al is telling the truth, then Bob and Carl are lying. This means that Bob would be speaking the truth if he said he did it. But then both Al and Bob did it and that's a contradiction.

If Bob is telling the truth, then Al and Carl are lying. This means that Al is speaking the truth if he said he did not do it and that Carl would be speaking the truth if he said that Al did do it. That is also a contradiction.

If Carl is telling the truth then Al didn't do it. Also, Al and Bob are lying. So, Al would be speaking the truth if he said that he didn't do it. That is consistent with Carl. Bob would be speaking the truth if he said that he did it. So Bob did it.

A more difficult riddle is this:

Each resident in the town of Xenia always lies or always tells the truth. Yanni and Zelda live in Xenia.

Yanni says: "Exactly one of us is lying."

Zelda says: "Yanni is telling the truth."

Determine who is telling the truth and who is lying.

Assume that Yanni is telling the truth. Then Zelda is lying but that means that Yanni is also lying and that leads to a contradiction.

That means that Yanni is a liar so "Exactly one of us is lying" is false. This leads to the conclusion that either both are truth tellers or both are liars.

Suppose both are telling the truth. But this can't be because we know that Yanni is a liar.

The last possibility is that both are liars and are lying. Then Yanni's statement "Exactly one of us is lying" is false, and Zelda's statement "Yanni is telling the truth" is false. This outcome does not lead to any contradiction. In fact it is consistent, meaning that Yanni and Zelda are both liars.

***How and Why Does Indirect Proof Work?***

When it comes to indirect proof in geometry or algebra we will reason in ways similar to the above problems, but we will be more formal and structured.

It was either in geometry or philosophy that the human race discovered that it could reason in very precise, logical ways. This discovery naturally led to an extensive analysis of what constituted valid reasoning and what constituted fallacious reasoning. The point was to establish ways by which one could start with a true statement and reason with confidence to a true statement. Let *TS* stand for a true statement, *FS* stand for a false statement, and *let the arrow stand for valid reasoning*.

Then it must be the case that $TS \to TS$, and **it must be impossible for $TS \to FS$, since otherwise we could never trust our reason**. What are the other possibilities — could we have $FS \to FS$ and $FS \to TS$? Consider the following examples:

| $TS \to TS$ | $FS \to FS$ | $FS \to TS$ |
|---|---|---|
| $1 = 1$ | $1 = 2$ | $1 = 2$ |
| $1+3 = 1+3$ | $1+3 = 2+3$ | $2 = 1$ |
| $4 = 4$ | $4 = 5$ | $1+2 = 2+1$ |
| | | $3 = 3$ |

We see that clearly, both $FS \to FS$ and $FS \to TS$ are possible; but while we may have expected to see $FS \to FS$, the third example ( $FS \to TS$ ) is somewhat surprising. In the case of $FS \to TS$, we reasoned this way: Although 1 = 2 is false, it is nonetheless true that if $a = b$ then $b = a$, and on that basis we obtain the second step. Then we add equals to both sides and arrived at a true result. So one can reason validly from a false statement to a false statement **or** from a false statement to a true statement.

It is a pity that many students fail to understand two major implications of the above analysis. The first major implication is this: Let *P* be a statement whose validity we are trying to prove, i.e., we don't know if *P* is true. Suppose we reason from *P* to a true statement; that is, we have shown $P \to TS$. Does that mean that *P* is true? The answer, to be blunt, is not necessarily. The reason? Both $TS \to TS$ and $FS \to TS$ are possible.

The second major implication is that if one starts with a statement *P*, reasons correctly, and obtains a false statement, then *P* has to be false because it is impossible to reason from a true statement to a false statement. Thus, one way to prove that *P* is true is to start with its opposite, $\sim P$, and show that $\sim P$ leads to a false conclusion. That means that $\sim P$ is false. Therefore *P* is true. Such a proof is called **proof by contradiction, an indirect proof, or proof by reductio ad absurdum**. In an indirect proof we start with, or assume, the opposite or negation of what we want to prove and show that the negation leads to a contradiction.

### ***Algebraic Examples***

Algebraic examples are often easier to follow at first than geometric. Here's an algebraic example:

Prove: For $a, b \geq 0$, $\dfrac{a+b}{2} \geq \sqrt{ab}$.

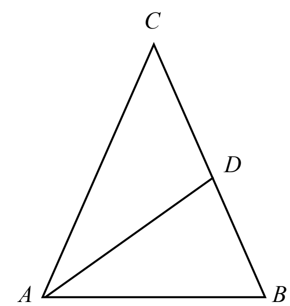
Proof: Start with or suppose that $\dfrac{a+b}{2} < \sqrt{ab}$. Then $a + b < 2\sqrt{ab}$. Squaring gives $a^2 + 2ab + b^2 < 4ab$, making $a^2 - 2ab + b^2 < 0$. Then $(a-b)^2 < 0$ and that is a contradiction since any real number squared is positive or 0. Hence, $\dfrac{a+b}{2} < \sqrt{ab}$ is false since it led to a contradiction, making $\dfrac{a+b}{2} \geq \sqrt{ab}$ true.

Here's a second example: prove that the difference between a rational number $x = \frac{a}{b}$ and an irrational number $y$ must be irrational.

Assume that the difference is rational. Then $y - \frac{a}{b} = \frac{m}{n} \rightarrow y = \frac{a}{b} + \frac{m}{n} = \frac{an + bm}{bn}$. Since the integers are closed under addition and multiplication, $\frac{an + bm}{bn}$ is rational and that contradicts the given, then the supposition that the difference is rational is false. Thus the difference is irrational.

***Finally, a Geometric Example***

Here is a geometric example of an indirect proof in paragraph form.




Given: In $\Delta ABC$, $AB \neq AC$ and $\overline{AD}$ is a median.

Prove: $\overline{AD}$ cannot be perpendicular to $\overline{BC}$.

Proof: Either $\overline{AD}$ is perpendicular to $\overline{BC}$ or it is not. What happens if we assume that $\overline{AD} \perp \overline{BC}$? In that case, $\angle ADB$ and $\angle ADC$ would both be right angles and hence congruent. We are given that $\overline{AD}$ is a median, so $\overline{CD} \cong \overline{DB}$. Since $\overline{AD} \cong \overline{AD}$ then $\Delta ADC \cong \Delta ADB$ by SAS. By CPCTC, we then have $\overline{AB} \cong \overline{AC}$, but this would contradict our given, that $AB \neq AC$. Therefore our assumption that $\overline{AD}$ is perpendicular to $\overline{BC}$ is false; $\overline{AD}$ cannot possibly be perpendicular to $\overline{BC}$.

***Be Intriguing, use Indirect Proof and Produce Paradoxes***:

Indirect proof can lead interestingly enough to paradoxes: consider a completely isolated village—no one enters, no one leaves. There is 1 barber. He shaves all and only those who don't shave themselves. Does he shave himself?

If he doesn't, then he does. If he does, then he doesn't.

***The Power of Reason***:

Finally, how powerful is reason. Can we prove God's existence?

> Philosophers and theologians have often wondered about the power of human reason. What can we know by reasoning alone? Anselm of Canterbury (1033 – 1109) wondered if we could prove that God existed by reason alone. Here is his proof by contradiction that God exists:

> Our idea of God is that he is a being greater than anything else that can be conceived. Existence in both reality and imagination is greater than existence solely in the imagination. If God did not exist in reality, then I could conceive of a being with all the attributes of God plus existence in reality. Such a being would be greater than God and that would be a contradiction, contradicting our idea of God. So God must exist in reality. Hmm . . .

**Origins of Indirect Proof**:

We see no evidence of the use of indirect proof among early mathematicians in the Sumerian, Babylonian, Indian, Egyptian, or Chinese civilizations. It seems to emerge in Greece. How does one discover an entirely new way of thinking, how does it come to be valued, explored, and used?

The record is not clear. It may have arisen among philosophers as a way of arguing—show that the other person's idea leads to absurd conclusions. Perhaps with Xenophanes of Colophon (c.570 – c.475 BC) or Parmenides (c.515 BC – ) but there is no evidence of their using it in mathematics. In some of Plato's dialogues we see a form of argument called dialectic which mimics indirect proof. For example, suppose that Theaetetus says that all knowledge is perception. Then Socrates will ask a series of questions whose answers eventually lead Theaetetus to a logical conclusion that he finds himself disagreeing with. He then concedes that he was wrong.

Zeno (fl 450 BC) used a form of indirect proof in establishing his paradoxes. The question is: are space and time infinitely divisible. Suppose they are not. That is, suppose they consist of small, indivisible parts. In *Arrow* he argues that therefore an arrow must be at rest over each time interval and since it can't move between time intervals, motion is an illusion and is impossible. This is absurd so space and time must be infinitely divisible. But Zeno was a clever soul and so he also proved that space and time consisted of small, indivisible parts. Suppose space and time do not consist of small, indivisible parts. Then they must be infinitely divisible. In *Achilles* he argues that therefore a faster runner could never overtake a slower runner (ie tortoise) who starts with a lead because the pursuer is always reaching the point that the slower runner has already left and this situation will continue indefinitely since space and time are infinitely sub-divisible, so that motion is again impossible and since that is absurd, space and time must consist of small, indivisible parts.

The first mention of the use of proof by contradiction in mathematics is the story about Hippasus of Metapontum proving to his fellow Pythagoreans in the fifth century BC that the diagonal of a square is incommensurable with its sides, meaning that if the sides are rational, then the diagonal is irrational. His proof seems to have been something like this:

Suppose that $\sqrt{2}$ is rational. Then it can be written as the ratio of two integers, in particular as the ratio of two integers with no common factors. So let $\sqrt{2} = \frac{a}{b}$ giving $2b^2 = a^2$. Note that an odd integer squared is odd and an even integer squared is even. Since $a^2$ equals an even number, $a$ itself must be even. This implies that $a = 2k$ for some integer $k$. Substituting back into $2b^2 = a^2$ and simplifying gives $2b^2 = (2k)^2 \rightarrow b^2 = 2k^2$. Since $b^2$ is even, then $b$ must be even, but that has led us to a contradiction since $a$ and $b$ have a common factor of 2 that contradicts the given that they have no common factors.

This can be proved differently, yet indirectly: Assume that $\sqrt{2} = \frac{a}{b}$ where $a$ and $b$ have no common factors. Then $2b^2 = a^2$. Consider the units digit in $a^2$. It must be 0, 1, 4, 5, 6, or 9. Consider the units digit in $2b^2$. It must be 0, 2, or 8. For $2b^2$ to equal $a^2$, they must both end in 0. But then they have a common factor of 5 or 2 or 10.

This second proof has legs. Use it to prove that $\sqrt{3}$ is irrational: Assume that $\sqrt{3} = \frac{a}{b}$ where $a$ and $b$ have no common factors. Then $3b^2 = a^2$. Consider the units digit in $a^2$. It must be 0, 1, 4, 5, 6, or 9. Consider the units digit in $3b^2$. It must be 0, 2, 3, 5, 7, or 8. For $3b^2$ to equal $a^2$, they must both end in 0 or 5. But then they have a common factor of 5 and that contradicts the given.

***Precision, truth, and indirect proof***:

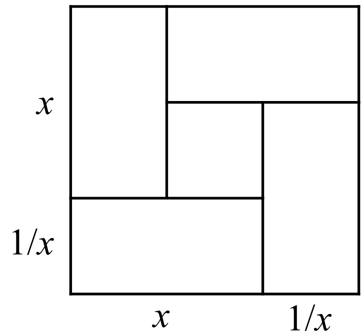
For indirect proof to occur there had to be a transition to precision in language. Something similar happened with area. The record suggests that the Babylonians measured area in the following way: This field takes 2 bags of seed to sow while this takes 4. Then they moved to an abstract, precise way of calculation: length times width. Medieval England measured land by the number of pigs it supported.

To do indirect proof, one to think of statements as either 100% true or false. This represents quite a change because usually we think that an idea is somewhat true.

Thales, circa 600 BC is credited with the first proofs: the base angles of an isosceles triangle are equal, the diameter of a circle divides it into two equal parts, nothing difficult. But there must have been an emphasis on precision in language and a sense that our words referred to an ideal world where something was either true, completely true, or false, completely false. Prior to Thales, we suspect that proofs were diagrammatic. The tradition is still being carried on--see "Proofs without Words" in *Mathematics Magazine*. Here are several:

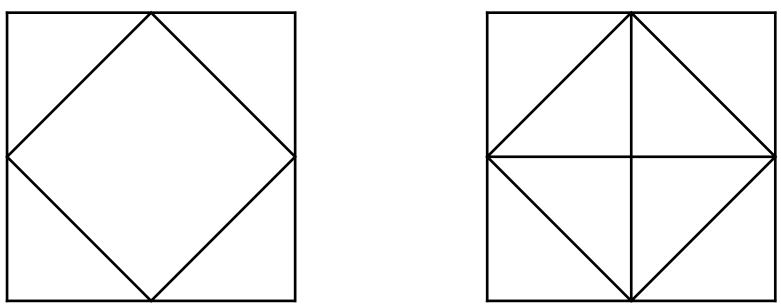
Prove that the sum of a positive number and its reciprocal is greater than or equal to 2:

BEHOLD



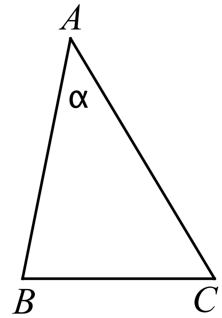

Since the area of each rectangle is 1, then $\left(x+\frac{1}{x}\right)^2 \geq 4 \rightarrow x+\frac{1}{x} \geq 2$ .

We have in Plato's Meno (c. 427 – 347 BC) what may be a description of early diagrammatic proofs: Socrates is showing a young man that the area of a square whose side is the hypotenuse of an isosceles right triangle is twice the area of the square whose side is a leg of that triangle.

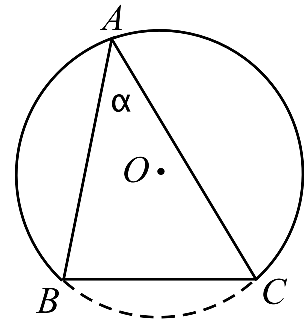


Finally, the Greeks were fascinated by locus problems. That is, given a set of conditions on a point, what path does the point take or what curve do all points satisfying those conditions lie on. They often used indirect proof and the trichotomy principle to prove that the locus was what they thought it was. For example, if they thought the point lay on a circle, they would first assume that it lay inside the circle and show that that led to a contradiction. Then they would assume that it lay outside the circle and show that that, too, led to a contradiction. Then the only other possibility is that the point lies on the circle. Here is a worked example done in the way that Euclid wrote:

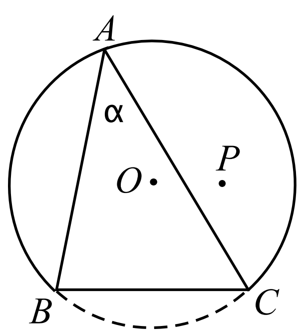
Consider all points $A$ that make a given angle $\alpha$ with the line segment $\overline{BC}$.

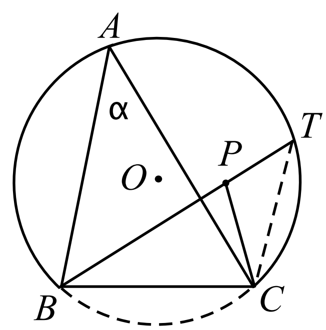



I claim that all points $A$ lie on the major arc of the circle $O$ circumscribing $\Delta ABC$ as shown by the solid arc:

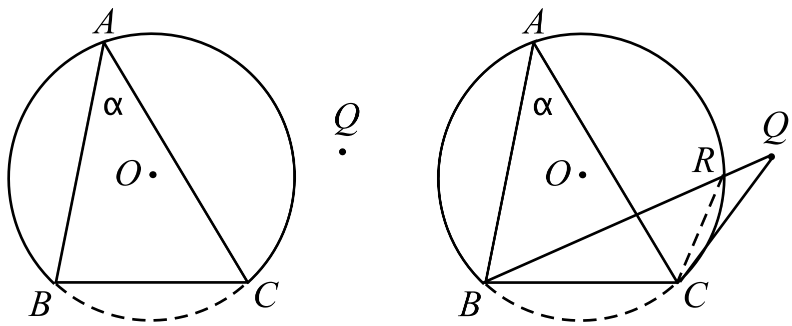



For if not, suppose that the point, call it $P$, lies inside the circle. Then draw the segment from $B$ through $P$ intersecting the circle at $T$. Since we know that inscribed angles are equal, $m\angle BTC = \alpha$. By the Exterior Angle Theorem, $m\angle BPC > m\angle T$ and therefore that contradicts the claim that the angle at a point $P$ inside the circle subtends the same angle as $\alpha$.



Now suppose that the point, call it *Q*, lies outside the circle. Then draw the segment from *B* to *Q* intersecting the circle at *R*. Since we know that inscribed angles are equal, $m\angle BRC = \alpha$. By the Exterior Angle Theorem, $m\angle BRC > m\angle Q$ and therefore that contradicts the claim that the angle at a point *Q* outside the circle subtends the same angle as $\alpha$.




Since *A* either lies inside, outside, or on the circle, and we have shown it can't lie inside or outside, it must, therefore, lie on the circle.

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